

# Elements of harmonic analysis, 3

Stephen William Semmes  
Rice University  
Houston, Texas

These informal notes are based on a course given at Rice University in the spring semester of 2004, and much more information can be found in the references.

## Locally compact abelian topological groups

Let  $A$  be an abelian group. Thus  $A$  is a set equipped with a binary operation  $+$  which is commutative and associative, there is an identity element  $0 \in A$  such that  $0 + a = a$  for all  $a \in A$ , and each  $a \in A$  has an inverse  $-a$  characterized by  $a + (-a) = 0$ . As basic examples, the integers, real numbers, and complex numbers are abelian groups under addition, and for each positive integer  $n$  we have the integers modulo  $n$ , a cyclic group with  $n$  elements.

Let us also assume that  $A$  is a topological space, which is to say that certain subsets of  $A$  are designated as open subsets. As usual one requires that the empty set and  $A$  itself are open subsets of  $A$ , that the intersection of any finite collection of open subsets is an open subset, and that the union of any collection of open subsets is an open subset. Once the open subsets are selected, the closed subsets are defined to be the complements in  $A$  of the open subsets. Various standard notions, such as continuity at a point of a mapping between two topological spaces, can be defined in terms of the open subsets through standard methods.

To say that  $A$  is a topological group means that the group structure and topology are compatible in a natural way. Specifically the group operation  $+$  should be continuous as a mapping from  $A \times A$  into  $A$ , and  $a \mapsto -a$  should be continuous as a mapping from  $A$  to itself. This implicitly uses the product topology on  $A \times A$  induced by the given topology on  $A$ , in which a subset of  $A \times A$  is open if it is the union of products of open subsets of  $A$ . It is customary to require that  $A$  be a Hausdorff topological space, which is equivalent in this setting to the requirement that  $\{0\}$  be a closed subset of  $A$ .

In any topological space a subset  $K$  is said to be compact if every open covering of  $K$  in the space admits a finite subcovering, i.e., if for every family

$\{U_\iota\}_{\iota \in I}$  of open subsets of the topological space such that

$$(1) \quad K \subseteq \bigcup_{\iota \in I} U_\iota$$

there is a finite collection  $\iota_1, \dots, \iota_l$  of indices in  $I$  such that

$$(2) \quad K \subseteq U_{\iota_1} \cup \dots \cup U_{\iota_l}.$$

A topological space is said to be locally compact if for each point  $x$  in the space there is an open subset  $W$  and a compact subset  $K$  of the space such that

$$(3) \quad x \in W \subseteq K.$$

A locally compact abelian topological group is an abelian topological group which is locally compact as a topological space. Of course local compactness at the identity element 0 implies local compactness at every point because group translations define homeomorphisms.

As in [3], it is simpler to say “LCA group” in place of locally compact abelian topological group. The integers and the integers modulo  $n$  are natural examples of LCA groups equipped with their discrete topologies, in which every subset is considered to be open. For the real and complex numbers one can use their standard topologies, induced by the usual Euclidean metrics, to get LCA groups. One can also consider the nonzero complex numbers using multiplication as the group operation and the usual topology. If one takes the complex numbers with modulus 1 using multiplication as the group operation and the usual topology, one gets a compact LCA group.

Fix an integer  $n \geq 2$ , and consider the group consisting of sequences  $x = \{x_j\}_{j=1}^\infty$  such that each  $x_j$  is an integer modulo  $n$ . One might as well say that each  $x_j$  is an integer such that  $0 \leq x_j \leq n-1$ , and the sum of two elements  $x, y$  in the group is defined by adding each term modulo  $n$ . If  $x$  is an element of the group and  $l$  is a positive integer, then the  $l$ th standard neighborhood around  $x$  is defined to be the set of  $y$  in the group such that  $x_j = y_j$  when  $1 \leq j \leq l$ . This leads to a topology on the space in which a subset of the group is open if for each point  $x$  in the subset there is a standard neighborhood around  $x$  which is contained in the subset. Well known results in topology imply that this space is compact with respect to this topology, and in fact it is homeomorphic to the Cantor set.

It is easy to see that this example defines a compact LCA group. Namely, the group operations are continuous with respect to the topology just defined. This is a nice example where the topological dimension is equal to 0, which is to say that the space is totally disconnected, with no connected subsets with at least two elements. At the same time the topology is not the discrete topology.

Let  $A$  be a LCA group. A basic object of interest associated to  $A$  is a translation-invariant integral, which is a linear mapping from the vector space of complex-valued continuous functions  $f(x)$  on  $A$  with compact support into the complex numbers such that the integral of  $f(x+a)$  is equal to the integral

of  $f(x)$  for all  $a \in A$ , the integral of a real-valued function is a real number, the integral of a nonnegative real-valued function is a nonnegative real number, and the integral of  $f$  is positive if  $f$  is a nonnegative real-valued continuous function on  $A$  with compact support such that  $f(x) > 0$  for some  $x \in A$ . In the examples described earlier such an integral can be defined explicitly, in terms of sums, classical Riemann integrals, or simple generalizations of Riemann integrals for the spaces of sequences modulo  $n$ . A general theorem states that any LCA group  $A$  has such an invariant integral, and that this integral is unique except for multiplying it by a positive real number.

Let  $A$  be an LCA group. By a character on  $A$  we mean a continuous group homomorphism from  $A$  into the group of complex numbers with modulus 1 with respect to multiplication. Sometimes one may wish to consider unbounded characters more generally, which are continuous homomorphisms from  $A$  into the nonzero complex numbers with respect to multiplication. Note that any bounded subgroup of the nonzero complex numbers with respect to multiplication is contained in the complex numbers with modulus 1, as one can easily verify. Thus a bounded continuous homomorphism from  $A$  into the group of nonzero complex numbers is a character, and in particular every continuous homomorphism from  $A$  into the nonzero complex numbers is a character when  $A$  is compact.

If  $f(x)$  is a complex-valued continuous function on  $A$  with compact support, or an integrable function more generally, one can define its Fourier transform  $\widehat{f}(\phi)$  by saying that if  $\phi$  is a character on  $A$ , then  $\widehat{f}(\phi)$  is the integral of  $f$  times the complex conjugate of  $\phi$ , using a fixed invariant integral on  $A$  as discussed previously. If  $A$  is not compact, then one can extend this to a Fourier–Laplace transform by allowing unbounded characters, at least when  $f$  has compact support or sufficient integrability properties. For bounded characters one has the usual inequality which states that  $|\widehat{f}(\phi)|$  is less than or equal to the integral of  $|f|$ .

Many classical aspects of Fourier analysis work in this setting. A basic point is that the Fourier transform diagonalizes translation operators, which means that if  $a \in A$  and  $f(x)$  is a continuous function on the group with compact support, or an integrable function on the group, then the Fourier transform of  $f(x - a)$  at the character  $\phi$  is equal to  $\overline{\phi(a)}$  times the Fourier transform of  $f(x)$  at  $\phi$ . One can also define convolutions in the usual way, using the invariant integral on  $A$ , and the Fourier transform of a convolution is equal to the product of the corresponding Fourier transforms.

## References

- [1] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton Mathematical Series 4, Princeton University Press, 1941.
- [2] E. Hewitt and K. Ross, *Abstract Harmonic Analysis I*, second edition, Springer-Verlag, 1979.

- [3] W. Rudin, *Fourier Analysis on Groups*, Wiley, 1962.
- [4] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Mathematical Series **32**, Princeton University Press, 1971.